ESSENTIAL TRIGONOMETRY WITHOUT GEOMETRY PART II: GENERALIZATIONS AND APPLICATIONS

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Abstract.–In a previous paper (Gresham et al. 2019), the properties, theorems, and identities of the sine and cosine functions were developed using only analytical methods and without geometric constructions. We follow those results and use them to develop generalizations of the key theorems of trigonometry, again using purely analytical methods. We conclude with a connection to the Law of Conservation of Energy in physics.

Keywords: classical analysis, trigonometry

In Gresham et al. (2019) a rigorous approach to the basic trig functions was developed where sin(x) was defined to be the solution of the 2nd order initial value problem (IVP)

$$f'' + f = 0, \text{ and } f(0) = 0, \ f'(0) = 1$$

and cos(x) was defined to be the derivative of sin(x). Here we extend this technique to define generalized trigonometric functions.

Definition A generalized sine function, or **sinusoid**, is defined to be a nonconstant differentiable function which is a solution of the linear homogeneous differential equation

$$f''(x) + k^2 f(x) = 0 \text{ and } f(0) = a, f'(0) = b$$

for some $k \neq 0$.

The condition that $k \neq 0$ ensures that f is not a linear function. The Existence and Uniqueness Theorem for Linear Homogeneous IVPs with constant coefficients (Nagle et al. 2008) tells us that such a solution exists, is unique, and has domain $(-\infty, \infty)$. In particular, it must be the case that for each real number t we must have $f(t) \neq 0$ or $f'(t) \neq 0$. If there is a value t where $f(t)$ and $f'(t)$ are both 0, then the

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solution will be the identically constant 0 function. Therefore the function f is nonconstant if and only if a and b are not both 0.

The following theorems contains elementary consequences of the definition.

Theorem If f is a sinusoid and $c \neq 0$ then $f(x + c)$, $f(c\,x)$, and $c\,f(x)$ are sinusoids.

Proof: First note that since f is nonconstant, these three transformations of f are nonconstant. It remains to show that they satisfy the differential equation for a sinusoid. Since f is a sinusoid, there is a nonzero constant k such that

$$f''(x) + k^2 f(x) = 0$$

for all x in $(-\infty, \infty)$. It suffices to show that

$$g''(x) + k^2 g(x) = 0$$

where g is one of the above transformations. We consider each case separately.

Case I. Let $g(x) = f(x+c)$. Then $g'(x) = f'(x+c)$ and $g''(x) = f''(x+c)$. It follows that

$$g''(x) + k^2 g(x) = f''(x+c) + k^2 f(x+c) = 0$$

and the definition of a sinusoid is satisfied.

Case II. Let $g(x) = f(cx)$.

Then $g'(x) = cf'(cx)$ and $g''(x) = c^2 f''(cx)$. It follows that

$$\begin{aligned} g''(x) &= c^2 f''(cx) \\ &= c^2(-k^2 f(cx)) \\ &= -c^2 k^2 f(cx) \\ &= -(ck)^2\, g(x). \end{aligned}$$

Therefore

$$g''(x) + (ck)^2 g(x) = 0.$$

Since $c \neq 0$, we must have $ck \neq 0$, and the definition of a sinusoid is satisfied.

Case III. Let $g(x) = cf(x)$. Then $g''(x) = cf''(x)$ and

$$\begin{aligned} g''(x) &= cf''(x) \\ &= c(-k^2 f(x)). \end{aligned}$$

It follows that

$$\begin{aligned} g''(x) + k^2 g(x) &= -ck^2 f(x) + k^2 c f(x) \\ &= 0 \end{aligned}$$

and the definition of a sinusoid is satisfied. ■

Theorem If *f* is a sinusoid then f' is a sinusoid.

Proof: We first show that f' satisfies the differential equation condition of the definition.

$$\begin{aligned} (f')'' + k^2 f' &= (f'')' + k^2 f' \\ &= (-k^2 f)' + k^2 f' \\ &= 0. \end{aligned}$$

Note that the initial conditions for f' at 0 are $f'(0) = b$ and $f''(0) = -k^2 f(0) = -k^2 a$. These are not both 0 since a and b are not both 0 and the number $k \neq 0$. ■

The following are consequences of this theorem.

Corollary If *f* is a sinusoid, then so is the *n*th derivative of *f*, denoted $f^{[n]}$.

Corollary If *f* is a sinusoid, then

$$f^{[2m]}(x) = (-1)^m k^{2m} f(x)$$

and

$$f^{[2m+1]} = (-1)^m k^{2m} f'(x).$$

The proofs of these are straightforward.

We denote a sinusoid $f(x)$ with the notation

$$f(x) = \sin_{k,a,b}(x)$$

where it is understood that $k \neq 0$, and that $\sin_{k,a,b}(0) = a$, and $\sin'_{k,a,b}(0) = b$, are not both 0.

We define its derivative to be

$$\cos_{k,a,b}(x) = \frac{d}{dx}\sin_{k,a,b}(x).$$

We will see that

$$\cos_{k,a,b}(x) = \sin_{k,b,-k^2a}(x).$$

In particular, the standard sine and cosine functions are the special cases

$$\sin(x) = \sin_{1,0,1}(x) \text{ and } \cos(x) = \sin_{1,1,0}(x).$$

Key theorems.–The two key theorems of plane trigonometry are the Pythagorean Identity and the Sine Sum Identity. With these two theorems one can develop all the other identities and values of special angles. In Gresham et al. (2019) it was shown that these can be rigorously proved without geometric constructions. Here we present a generalization of these two identities.

Theorem *Generalized Pythagorean Identity*

If $\sin_{k,a,b}(x)$ is a sinusoid then for all x

$$\left(k\sin_{k,a,b}(x)\right)^2 + \left(\cos_{k,a,b}(x)\right)^2 = c^2$$

where $c^2 = (ka)^2 + b^2$

Proof: We write $\sin_{k,a,b}(x) = f(x)$, $\cos_{k,a,b}(x) = f'(x)$ and consider

$$\begin{aligned}\frac{d}{dx}\left(\left(kf(x)\right)^2 + \left(f'(x)\right)^2\right) &= 2k^2 f(x) f'(x) + 2f'(x) f''(x) \\ &= 2k^2 f(x) f'(x) + 2f'(x)\left(-k^2 f(x)\right) \\ &= 0.\end{aligned}$$

The function $f(x)$ has a second derivative for all real numbers x and so $(kf(x))^2 + (f'(x))^2$ satisfies the conditions of the Mean Value Theorem.

Its derivative is 0 for all x, and so $(kf(x))^2 + (f'(x))^2$ must be a constant. Because $k \neq 0$, and $f(0) = a$, and $f'(0) = b$, with a and b are not both 0, this constant is

$$(kf(0))^2 + (f'(0))^2 = (ka)^2 + b^2 = c^2 > 0. \qquad \blacksquare$$

We note that using sinusoids in parametric equations of the form

$$x(t) = cos_{k,a,b}(t) \text{ and } y(t) = sin_{k,a,b}(t)$$

will produce the graph of an ellipse.

We now turn our attention to the relationship between general sinusoids and the basic sine and cosine functions.

Theorem *General Representation of a Sinusoid*

Sinusoids can be represented in terms of the standard sine and cosine functions as follows:

$$\sin_{k,a,b}(x) = a\cos(kx) + \frac{b}{k}\sin(kx).$$

Proof: Let $f(x) = \sin_{k,a,b}(x)$. We know that f is the solution of the IVP

$$k^2 f(x) + f''(x) = 0,\ k \neq 0, f(0) = a, f'(0) = b, a^2 + b^2 > 0.$$

Let $g(x) = a\cos(kx) + \frac{b}{k}\sin(kx)$.

Note that $g(0) = a$ and since

$$g'(x) = -ak\, sin(kx) + b\, cos(kx).$$

we have $g'(0) = b$. Then

$$\begin{aligned} g''(x) &= -ak^2 \cos(kx) - bk\, sin(kx) \\ &= -k^2 \left(\text{acos}(kx) + \frac{b}{k} \sin(kx) \right) \\ &= -k^2 g(x). \end{aligned}$$

Therefore *f* and *g* satisfy the same IVP and by uniqueness must be equal. ■

Using this General Representation theorem, we can produce a representation for $\cos_{k,a,b}(x)$.

Corollary $\cos_{k,a,b}(x) = \sin_{k,b,-k^2 a}(x)$.

Proof:

$$\begin{aligned} \cos_{k,a,b}(x) &= \frac{d}{dx}\sin_{k,a,b}(x) \\ &= \frac{d}{dx}\left(a\cos(kx) + \frac{b}{k}\sin(kx) \right) \\ &= b\cos(kx) - ka\sin(kx) \\ &= \sin_{k,b,-k^2 a}(x). \end{aligned}$$

■

The Sine Sum formula, whose non-geometric proof is included in Gresham et al. (2019),

$$\sin(x + y) = \sin(x)\cos(y) + \cos(x)\sin(y)$$

serves as a key identity in the development of classical trigonometry.

To prepare for a generalization of this identity in terms of $\sin_{k,a,b}(x)$ and $\cos_{k,a,b}(x)$, we next consider a representation of $sin_{k,a,b}(x)$ as a transformation of the classical sine function.

Lemma *Transformation Representation* Let z be the complex number $\frac{b}{k} + ai$. Then

$$\sin_{k,a,b}(x) = |z| \sin(kx + \arg z)$$

where $\arg z$ is the argument of the complex number z.

Proof: With $z = \frac{b}{k} + ai$, we have

$$\sin(\arg z) = \frac{a}{|z|} \text{ and } \cos(\arg z) = \frac{\frac{b}{k}}{|z|}.$$

It follows then that

$$\begin{aligned} |z| \sin(kx + \arg z) &= |z|(\sin(kx)\cos(\arg z) + \cos(kx)\sin(\arg z)) \\ &= |z| \left(sin(kx) \frac{\frac{b}{k}}{|z|} + cos(kx) \frac{a}{|z|} \right) \\ &= \frac{b}{k} sin(kx) + a \cos(kx) \\ &= \sin_{k,a,b}(x). \end{aligned}$$

■

Using this result and the Sine Sum formula, we can write the Sinusoid Sum formula.

Theorem *Sinusoid Sum Formula*

If $f(x) = \sin_{k,a,b}(x)$ is a sinusoid and $z = \frac{b}{k} + ai$, then

$$f(x+y) = \frac{1}{k|z|}\left(f\left(x - \tfrac{\arg z}{k}\right)f'(y) + f'\left(x - \tfrac{\arg z}{k}\right)f(y)\right)$$

Proof: In this proof we will view *y* as an arbitrary fixed real constant. We will show that the left and right sides of the equation above are sinusoids in *x* with the same initial conditions.

Observe that since *f* is a sinusoid, the left side of the equation, a horizontal transformation of *f*, is a sinusoid and must satisfy the condition

$$\frac{d^2}{dx^2} f(x+y) = -k^2 f(x+y).$$

To simplify our analysis of the right side, let *g* (*x*) represent the right side, again viewing *y* as a constant:

$$g(x) = \frac{1}{k|z|}\left(f\left(x - \tfrac{\arg z}{k}\right)f'(y) + f'\left(x - \tfrac{\arg z}{k}\right)f(y)\right).$$

Now we find the first and second derivatives of the right side, treating *y* as a constant.

$$\begin{aligned} g'(x) &= \frac{1}{k|z|}\left(f'\left(x - \tfrac{\arg z}{k}\right)f'(y) + f''\left(x - \tfrac{\arg z}{k}\right)f(y)\right) \\ &= \frac{1}{k|z|}\left(f'\left(x - \tfrac{\arg z}{k}\right)f'(y) - k^2 f\left(x - \tfrac{\arg z}{k}\right)f(y)\right) \end{aligned}$$

and

$$\begin{aligned}
g''(x) &= \frac{d}{dx} g'(x) \\
&= \frac{d}{dx}\frac{1}{k|z|}\left(f'\left(x-\tfrac{\arg z}{k}\right)f'(y) - k^2 f\left(x-\tfrac{\arg z}{k}\right)f(y)\right) \\
&= \frac{1}{k|z|}\left(f''\left(x-\tfrac{\arg z}{k}\right)f'(y) - k^2 f'\left(x-\tfrac{\arg z}{k}\right)f(y)\right) \\
&= \frac{1}{k|z|}\left(-k^2 f\left(x-\tfrac{\arg z}{k}\right)f'(y) - k^2 f'\left(x-\tfrac{\arg z}{k}\right)f(y)\right) \\
&= -k^2\left(\frac{1}{k|z|}\left(f\left(x-\tfrac{\arg z}{k}\right)f'(y) + f'\left(x-\tfrac{\arg z}{k}\right)f(y)\right)\right) \\
&= -k^2\left(\frac{1}{k|z|}\left(f\left(x-\tfrac{\arg z}{k}\right)f'(y) + f'\left(x-\tfrac{\arg z}{k}\right)f(y)\right)\right) \\
&= -k^2 g(x).
\end{aligned}$$

From this we see that g satisfies the DE for a sinusoid. What are the initial values $g(0)$ and $g'(0)$? We first do some preliminary calculations using the Transformation Representation Theorem previously proved. Note that

$$\begin{aligned}
f'(x) &= \frac{d}{dx}\sin_{k,a,b}(x) \\
&= \frac{d}{dx}\left(|z|\sin(kx+\arg z)\right) \\
&= k|z|\cos(kx+\arg z)
\end{aligned}$$

We will need these two evaluations:

$$\begin{aligned}
f\left(-\frac{\arg z}{k}\right) &= |z|\sin\left(k\cdot\left(-\frac{\arg z}{k}\right)+\arg z\right) \\
&= |z|\sin(0) \\
&= 0
\end{aligned}$$

and

$$\begin{aligned} f'\left(-\frac{\arg z}{k}\right) &= k|z|\cos\left(k\cdot\left(-\frac{\arg z}{k}\right)+\arg z\right) \\ &= k|z|\cos(0) \\ &= k|z|. \end{aligned}$$

Now we are ready to calculate $g(0)$ and $g'(0)$:

$$\begin{aligned} g(0) &= \frac{1}{k|z|}\left(f\left(0-\tfrac{\arg z}{k}\right)f'(y)+f'\left(0-\tfrac{\arg z}{k}\right)f(y)\right) \\ &= \frac{1}{k|z|}\left(0\cdot f'(y)+k|z|f(y)\right) \\ &= f(y) \\ &= f(0+y) \end{aligned}$$

and

$$\begin{aligned} g'(0) &= \frac{1}{k|z|}\left(f'\left(0-\tfrac{\arg z}{k}\right)f'(y)-k^2 f\left(0-\tfrac{\arg z}{k}\right)f(y)\right) \\ &= \frac{1}{k|z|}\left(k|z|f'(y)-k^2\cdot 0\cdot f(y)\right) \\ &= f'(y) \\ &= f'(0+y). \end{aligned}$$

Since $f(x+y)$ is a sinusoid in x, $f(0+y)$ and $f'(0+y)$ are not both 0. Then $f(x+y)$ and $g(x)$ are both sinusoids in x with the same initial conditions, and therefore by Uniqueness they are equal. ■

Note Letting $f(x) = \sin_{k,a,b}(x)$, $f'(x) = \cos_{k,a,b}(x)$, and $z = \frac{b}{k} + ai$, the Sinusoid Sum formula can be written as

$$\sin_{k,a,b}(x+y) = \frac{1}{k|z|}\left(\sin_{k,a,b}\left(x-\tfrac{\arg z}{k}\right)\cos_{k,a,b}(y)+\cos_{k,a,b}\left(x-\tfrac{\arg z}{k}\right)\sin_{k,a,b}(y)\right).$$

Corollary The sine and cosine sum formulas are special cases of the General Sinusoid Sum Formula.

Proof: If $k = 1$, $a = 0$, and $b = 1$, then $f(x) = \sin(x)$ and $z = 1+0i$. The General Sum Formula reduces to

$$\begin{aligned}\sin(x+y) &= \frac{1}{k\cdot|z|}\left(f\left(x-\tfrac{\arg z}{k}\right)f'(y)+f'\left(x-\tfrac{\arg z}{k}\right)f(y)\right)\\ &= \frac{1}{1\cdot|1|}\left(\sin\left(x-\tfrac{0}{1}\right)\cos(y)+\cos\left(x-\tfrac{0}{1}\right)\sin(y)\right)\\ &= \sin(x)\cos(y)+\cos(x)\sin(y).\end{aligned}$$

If $k = 1$, $a = 1$, and $b = 0$, then $f(x) = \cos(x)$ and $z = 0 + 1\,i$. The General Formula reduces to

$$\begin{aligned}\cos(x+y) &= \frac{1}{k\cdot|z|}\left(f\left(x-\tfrac{\arg z}{k}\right)f'(y)+f'\left(x-\tfrac{\arg z}{k}\right)f(y)\right)\\ &= \frac{1}{1\cdot|i|}\left(\cos\left(x-\tfrac{\pi/2}{1}\right)(-\sin(y))+\left(-\sin\left(x-\tfrac{\pi/2}{1}\right)\right)\cos(y)\right)\\ &= -\cos\left(x-\frac{\pi}{2}\right)\sin(y)-\sin\left(x-\frac{\pi}{2}\right)\cos(y)\\ &= -\cos\left(\frac{\pi}{2}-x\right)\sin(y)+\sin\left(\frac{\pi}{2}-x\right)\cos(y)\\ &= -\sin(x)\sin(y)+\cos(x)\cos(y).\end{aligned}$$

■

We now observe that with additional conditions, namely that the function f be analytic, the converse of the General Pythagorean Identity also holds.

Theorem If $f(x)$ is nonconstant, analytic on $(-\infty,\infty)$, and for all real x satisfies the Generalized Pythagorean Identity

$$(k\,f(x))^2 + (f'(x))^2 = c^2,\ \ c \neq 0,\ k \neq 0$$

where k and c are constants, then f is a sinusoid.

Proof: Differentiation of both sides gives

$$2\left(k^2 f(x)\right) f'(x) + 2f'(x) f''(x) = 0$$

so that

$$2f'(x)\left(k^2 f(x) + f''(x)\right) = 0.$$

Since f is nonconstant, $f'(d) \neq 0$ for some number d. Because f is analytic, f' is continuous and there exists an open interval S containing d such that $f'(x)$ is nonzero on S. On S it must be the case that

$$k^2 f(x) + f''(x) = 0.$$

The function f is a solution of the differential equation for a sinusoid. Since such solutions are unique, it must follow that f is the solution with domain $(-\infty, \infty)$. The function f is nonconstant and it must follow that $f(0)$ and $f'(0)$ are not both 0. Therefore the function f is a sinusoid.

■

Connection with the Law of Conservation of Energy.–Simple Harmonic Motion (SHM) is used to describe a system in which an object of mass m experiences a restoring force which is directly proportional to its displacement from its equilibrium position and there are no other forces such as damping forces or friction involved. The common applications are pendulums with small displacements or an oscillating mass on a spring. We will consider a system with a spring with spring constant k connected to an object of mass m with no other forces involved. We know from Hooke's Law that in such a system the restoring force is described by

$$F = -k\,x$$

where $x = x(t)$ is a one-dimensional vector which measures its displacement from the equilibrium position at time t.

In the SI system, the magnitude of this one-dimensional force vector is measured in Newtons (**N**), the spring constant $k > 0$ is measured in Newtons/meter(**N/m**), and the magnitude of displacement vector $x(t)$ is measured in meters. The spring potential energy (**SPE**), measured in Joules (**J**), in the system is

$$\text{SPE} = \frac{1}{2}kx^2 \text{ J}$$

and if v is the magnitude of the one-dimensional velocity vector of the mass, the kinetic energy (**KE**) in the system is

$$\text{KE} = \frac{1}{2}mv^2 \text{ J.}$$

Our goal here is two-fold: first to describe the position $x(t)$ of the mass at time t, and second, to describe the energy of the system. We begin with a solution for the differential equation given by Hooke's Law.

$$F(t) = -k\,x(t) \text{ with } x(0) = x_0 \text{ and } x'(0) = v_0 .$$

Since

$$F = ma = mx''(t)$$

we can write the differential equation for SHM

$$\begin{aligned} mx''(t) &= -kx(t) \\ mx''(t) + kx(t) &= 0 \\ x''(t) + \frac{k}{m}x(t) &= 0 \\ x''(t) + \omega^2 x(t) &= 0 \text{ where} \end{aligned}$$

$$\omega^2 = \frac{k}{m},\ x(0) = x_0 \text{ and } x'(0) = v_0.$$

From what we have seen earlier, the solution of this system is

$$x(t) = x_0 \cos\left(\omega t\right) + \frac{v_0}{\omega} \sin\left(\omega t\right) \text{ where } \omega = \sqrt{\frac{k}{m}}.$$

Next, we consider the connection between the Generalized Pythagorean Identity for this sinusoid and the energy of the system. We know that at any time t

$$\left(\omega\, x(t)\right)^2 + \left(x'(t)\right)^2 = c^2.$$

From substituting the initial conditions at $t = 0$,

$$\begin{aligned} c^2 &= \left(\omega x_0\right)^2 + v_0^2 \\ &= \omega^2 x_0^2 + v_0^2 \\ &= \frac{k}{m} x_0^2 + v_0^2. \end{aligned}$$

We now write this last expression in terms of the SPE and KE of the system.

$$\begin{aligned} \frac{k}{m} x_0^2 + v_0^2 &= \frac{2}{m}\left(\frac{1}{2} k x_0^2 + \frac{1}{2} m v_0^2\right) \\ &= \frac{2}{m}\left(SPE + KE\right). \end{aligned}$$

From the Law of Conservation of Energy we know that the total energy of the harmonic oscillator is constant. Therefore, the Generalized Pythagorean Theorem produces a constant which is twice the energy of the system (in Joules) divided by the mass (in kg) of the system. The energy of a system divided by its mass is called the **specific energy** of the system.

We can now state the result of this discussion.

Theorem When the motion of a mass attached to a spring in a frictionless system is described by a sinusoid, then the constant c^2 in the Generalized Pythagorean Identity for this sinusoid is twice the specific energy of the system.

Summary and Conclusions.–We have extended the theorems and identities of basic trigonometry using the definition of a generalized sine function, called a sinusoid, as the solution of a certain second order linear homogeneous differential equation. The sine function itself can be defined as a solution for particular initial values, and expressed in terms of a power series. The key theorems in this study are the Pythagorean Identity and the Sine Sum Identity, which we have stated in a more general form here. The Pythagorean Identity, in its generalized form, characterizes sinusoid functions as well as providing a connection to physics.

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